

# Charge Conjugation from Space-Time Inversion

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**Abstract** We show that the *CPT* group of the Dirac field emerges naturally from the *PT* and *P* (or *T*) subgroups of the Lorentz group.

**Keywords** *CPT* group · Space-time inversion · Lorentz group

## 1 *CPT* Group of the Dirac Field

In a recent paper [5], it was shown that the *CPT* group  $G_{\Theta}$  ( $\Theta$  for the product  $\mathbf{C} * \mathbf{P} * \mathbf{T}$  of the three operators:  $\mathbf{C}$ , charge conjugation;  $\mathbf{P}$ , space inversion;  $\mathbf{T}$ , time reversal) of the Dirac quantum field  $\hat{\psi}$ , was isomorphic to the direct product of the quaternion group  $Q$  and the cyclic group of two elements  $\mathbb{Z}_2$  i.e.

$$G_{\Theta} \cong Q \times \mathbb{Z}_2. \quad (1)$$

The product  $\mathbf{A} * \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are any of the operators  $\mathbf{C}$ ,  $\mathbf{P}$ ,  $\mathbf{T}$ , is given by  $(\mathbf{A} * \mathbf{B}) \cdot \hat{\psi} = (\mathbf{A}\mathbf{B})^\dagger \hat{\psi} (\mathbf{A}\mathbf{B})$ .  $Q$  consists of the elements  $1, \iota, \gamma, \chi$  and their negatives, with the group multiplication  $\iota^2 = \gamma^2 = \chi^2 = -1, \iota\gamma = -\gamma\iota = \chi, \gamma\chi = -\chi\gamma = \iota$  and  $\chi\iota = -\iota\chi = \gamma$ .  $\iota, \gamma$  and  $\chi$  are the three imaginary units defining the quaternion numbers. So  $G_{\Theta}$  is one of the nine non abelian groups of a total of fourteen groups with sixteen elements; only three of the non abelian groups have three generators. In particular,  $G_{\Theta} \cong DC_8 \times \mathbb{Z}_2$ , where  $DC_8$  is the dicyclic group of eight elements with generators  $x$  and  $y$ ; so, the generators of  $DC_8 \times \mathbb{Z}_2$  are  $(x, 1)$ ,  $(y, 1)$  and  $(1, z)$ , where  $z$  is the generator of  $\mathbb{Z}_2$  [1]. The isomorphism with  $Q$  is given by the correspondence  $x \rightarrow \iota$  and  $y \rightarrow \gamma$ .

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The table of  $G_\Theta$  is the following:

	<b>C</b>	<b>P</b>	<b>T</b>	<b>C*P</b>	<b>C*T</b>	<b>P*T</b>	$\Theta$	
<b>C</b>	1	<b>C*P</b>	<b>C*T</b>	<b>P</b>	<b>T</b>	$\Theta$	<b>P*T</b>	
<b>P</b>	<b>C*P</b>	-1	<b>P*T</b>	- <b>C</b>	$\Theta$	- <b>T</b>	- <b>C*T</b>	
<b>T</b>	<b>C*T</b>	- <b>P*T</b>	-1	- $\Theta$	- <b>C</b>	<b>P</b>	<b>C*P</b>	
<b>C*P</b>	<b>P</b>	- <b>C</b>	$\Theta$	-1	<b>P*T</b>	- <b>C*T</b>	- <b>T</b>	(2)
<b>C*T</b>	<b>T</b>	- $\Theta$	- <b>C</b>	- <b>P*T</b>	-1	<b>C*P</b>	<b>P</b>	
<b>P*T</b>	$\Theta$	<b>T</b>	- <b>P</b>	<b>C*T</b>	- <b>C*P</b>	-1	- <b>C</b>	
$\Theta$	<b>P*T</b>	<b>C*T</b>	- <b>C*P</b>	<b>T</b>	- <b>P</b>	- <b>C</b>	-1	

This table must be completed by adding to the first row and to the first column the negatives  $-\mathbf{C}$ ,  $-\mathbf{P}$ ,  $\dots$ ,  $-\Theta$ ,  $-1$ , and making the corresponding products. The isomorphism  $G_\Theta \rightarrow Q \times \mathbb{Z}_2$  is given by

$$\begin{aligned}
 1 &\mapsto (1, 1), \\
 \mathbf{C} &\mapsto (1, -1), \\
 \mathbf{P} &\mapsto (\iota, 1), \\
 \mathbf{T} &\mapsto (\gamma, 1), \\
 \mathbf{C*P} &\mapsto (\iota, -1), \\
 \mathbf{C*T} &\mapsto (\gamma, -1), \\
 \mathbf{P*T} &\mapsto (\chi, 1), \\
 \Theta &\mapsto (\chi, -1), \\
 -\mathbf{C} &\mapsto (-1, -1), \\
 -\mathbf{P} &\mapsto (-\iota, 1), \\
 -\mathbf{T} &\mapsto (-\gamma, 1), \\
 -\mathbf{C*P} &\mapsto (-\iota, -1), \\
 -\mathbf{C*T} &\mapsto (-\gamma, -1), \\
 -\mathbf{P*T} &\mapsto (-\chi, 1), \\
 -\Theta &\mapsto (-\chi, -1), \\
 -1 &\mapsto (-1, 1)
 \end{aligned}
 \tag{3}$$

where we identified  $\mathbb{Z}_2$  with the 0-sphere  $S^0 = \{1, -1\}$ .

### 2 Q as a Subgroup of SU(2)

With the correspondence

$$\iota \mapsto \rho_1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

$$\gamma \mapsto \rho_2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

$$\chi \mapsto \rho_3 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where  $\sigma_k, k = 1, 2, 3$  are the Pauli matrices,  $Q$  becomes isomorphic to a subgroup  $H$  of  $SU(2)$ . (Notice that  $\rho_k^\dagger = \rho_k^{-1}$  and  $\det \rho_k = +1$ ; also,  $H$  is not an invariant subgroup of  $SU(2)$ .)  $\mathbb{Z}_2$  is isomorphic to the center of  $SU(2)$ :  $\{\mathbb{I}, -\mathbb{I}\}$  with  $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , and so

$$G_\Theta \cong H \times (\text{center of } SU(2)). \tag{5}$$

Since  $SU(2)$  is the *universal covering group* of  $SO(3)$ :

$$SU(2) \xrightarrow{\Phi} SO(3) \tag{6}$$

with

$$\Phi \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} \operatorname{Re} z^2 - \operatorname{Re} w^2 & \operatorname{Im} z^2 + \operatorname{Im} w^2 & -2\operatorname{Re} zw \\ \operatorname{Im} z^2 - \operatorname{Im} w^2 & \operatorname{Re} z^2 + \operatorname{Re} w^2 & 2\operatorname{Im} zw \\ 2\operatorname{Re} z\bar{w} & 2\operatorname{Im} z\bar{w} & |z|^2 - |w|^2 \end{pmatrix}, \tag{7}$$

the number of finite subgroups of  $SU(2)$  is the same as the number of finite subgroups of  $SO(3)$ , which are: the cyclic groups  $C_n$ , for  $n = 1, 2, \dots$  ( $C_2 = \mathbb{Z}_2$ ); the dihedral groups  $D_k$ : symmetries of the rectangle for  $k = 2$  and of the regular polygons for  $k \geq 3$ ; and the rotational symmetry groups:  $T$  of the tetrahedron,  $O$  of the cube and the octahedron, and  $I$  of the dodecahedron and the icosahedron [6]. If  $A$  is a finite subgroup of  $SO(3)$ , then  $\tilde{A} = \Phi^{-1}(A)$  has twice the number of elements of  $A$ .

### 3 $\Phi(H)$ in $SO(3)$

$\Phi(H)$  has four elements and so the unique candidates for it are groups isomorphic to  $C_4$  and  $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , the Klein group. A simple application of (7) to the elements of  $H$  leads to

$$\Phi(H) = \{I, R_x(\pi), R_y(\pi), R_z(\pi)\} \tag{8}$$

with

$$R_x(\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_y(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \tag{9}$$

$$R_z(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the rotations in  $\pi$  around the  $x, y$  and  $z$  axes respectively, and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the unit matrix in  $SO(3)$ . It is immediately verified that the multiplication table of  $\Phi(H) \subset SO(3)$  is the same as for  $D_2$ , namely

$$\begin{array}{ccccc}
 & a & b & c & \\
 a & e & c & b & \\
 b & c & e & a & \\
 c & b & a & e & 
 \end{array} \tag{10}$$

with the correspondence  $a \mapsto R_x(\pi)$ ,  $b \mapsto R_y(\pi)$  and  $c \mapsto R_z(\pi)$ . Then we have

$$G_\Theta \cong \Phi^{-1}(D_2) \times \mathbb{Z}_2. \tag{11}$$

### 4 Parity and Time Reversal

Within the Lorentz group  $O(3, 1)$ , the transformations of *parity* and *time reversal* are given by the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

respectively. Together with the  $4 \times 4$  unit matrix  $E$  and their product  $PT$ , they lead to the subgroup of the Lorentz group with multiplication table

$$\begin{array}{cccc}
 & P & T & PT \\
 P & E & PT & T \\
 T & PT & E & P \\
 PT & T & P & E
 \end{array} \tag{13}$$

We call this group *the PT-group* of the Lorentz group. From (13) and (10), this group is isomorphic to  $D_2$ . On the other hand,  $P$  or  $T$  separately, together with the unit  $4 \times 4$  matrix  $E$ , give rise to the group  $\mathbb{Z}_2$ . Then we obtain the desired result:

$$G_\Theta \cong \Phi^{-1}(\langle\langle P, T \rangle\rangle) \times \langle\langle P \rangle\rangle \tag{14a}$$

or

$$G_\Theta \cong \Phi^{-1}(\langle\langle P, T \rangle\rangle) \times \langle\langle T \rangle\rangle. \tag{14b}$$

I.e.,  $G_\Theta$ , which in the present context is a group acting at the quantum field level that includes the charge conjugation operator for the electron field  $\hat{\psi}$ , emerges in a natural way from the pure space-time (and therefore classical)  $PT$ -group and its  $P$ - or  $T$ -subgroups.

### 5 Discussion

The above result suggests that the Minkowskian space-time structure of special relativity, in particular the unconnected component of its symmetry group, the real Lorentz group  $O(3, 1)$ , implies the existence of the  $CPT$  group as a whole, and therefore the existence of the charge conjugation transformation, and thus the proper existence of antiparticles. The

above conclusion (at least at the level of the electron field) is supported by the following consideration: even though the charge conjugation operation does not belong to the Lorentz group, at the level of the Dirac equation, the matrix

$$C = \pm i \gamma_2 \gamma_0 = \begin{pmatrix} 0 & 0 & 0 & \mp 1 \\ 0 & 0 & \pm 1 & 0 \\ 0 & \mp 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{pmatrix} \quad (15)$$

is an element of the Dirac algebra ( $\cong \mathbb{C}(4)$ ), which is the complexification of any of the two non isomorphic real Clifford algebras of Minkowski space-time:  $\mathbb{H}(2)$  ( $2 \times 2$  quaternion matrices for the metric  $\text{diag}(1, -1, -1, -1)$ ) and  $\mathbb{R}(4)$  ( $4 \times 4$  real matrices for the metric  $\text{diag}(-1, 1, 1, 1)$ ) [4].

Also, there is a sort of analogy between our result and the proof of the *CPT* theorem by Jost [3], who based his demonstration on the existence of the *PT* transformation as an element of the connected component of the complexification of the Lorentz group, and arguments of analytic continuation in field theory. See also Greenberg [2].

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